

A HILBERT-KUNZ CRITERION FOR SOLID CLOSURE IN DIMENSION TWO (CHARACTERISTIC ZERO)

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ABSTRACT. Let I denote a homogeneous R_+ -primary ideal in a two-dimensional normal standard-graded domain over an algebraically closed field of characteristic zero. We show that a homogeneous element f belongs to the solid closure I^* if and only if $e_{HK}(I) = e_{HK}((I, f))$, where e_{HK} denotes the (characteristic zero) Hilbert-Kunz multiplicity of an ideal. This provides a version in characteristic zero of the well-known Hilbert-Kunz criterion for tight closure in positive characteristic.

Mathematical Subject Classification (2000): 13A35; 13D40; 14H60

INTRODUCTION

Let (R, \mathfrak{m}) denote a local Noetherian ring or an \mathbb{N} -graded algebra of dimension d of positive characteristic p . Let I denote an \mathfrak{m} -primary ideal, and set $I^{[q]} = (f^q : f \in I)$ for a prime power $q = p^e$. Then the Hilbert-Kunz function of I is given by

$$e \longmapsto \lambda(R/I^{[p^e]}),$$

where λ denotes the length. The Hilbert-Kunz multiplicity of I is defined as the limit

$$e_{HK}(I) = \lim_{e \rightarrow \infty} \lambda(R/I^{[p^e]})/p^{ed}.$$

This limit exists as a positive real number, as shown by Monsky in [9]. It is an open question whether this number is always rational.

The Hilbert-Kunz multiplicity is related to the theory of tight closure. Recall that the tight closure of an ideal I in a Noetherian ring of characteristic p is by definition the ideal

$$I^* = \{f \in R : \exists c \text{ not in any minimal prime} : cf^q \in I^{[q]} \text{ for almost all } q = p^e\}.$$

For an analytically unramified and formally equidimensional local ring R the equation $e_{HK}(I) = e_{HK}(J)$ holds if and only if $I^* = J^*$ holds true for ideals $I \subseteq J$ (see [6, Theorem 5.4]). Hence $f \in I^*$ if and only if $e_{HK}(I) = e_{HK}((I, f))$. This is the Hilbert-Kunz criterion for tight closure in positive characteristic.

The aim of this paper is to give a characteristic zero version of this relationship between Hilbert-Kunz multiplicity and tight closure for R_+ -primary homogeneous ideals in a normal two-dimensional graded domain R . There

are several notions for tight closure in characteristic zero, defined either by reduction to positive characteristic or directly. We will work with the notion of solid closure (see [5]). In dimension two, the containment in the solid closure $f \in (f_1, \dots, f_n)^*$ means that the open subset $D(\mathfrak{m}) \subset \text{Spec } A$ is not an affine scheme, where $A = R[T_1, \dots, T_n]/(f_1T_1 + \dots + f_nT_n + f)$ is the so-called forcing algebra, see [1, Proposition 1.3].

The definition of the Hilbert-Kunz multiplicity in positive characteristic does not suggest at first sight an analogous notion in characteristic zero. However, a bridge is provided by the following result of [2], which gives an explicit formula for the Hilbert-Kunz multiplicity and proves its rationality in dimension two (the rationality of the Hilbert-Kunz multiplicity for the maximal ideal was also obtained independently in [10]).

Theorem 1. Let R denote a two-dimensional standard-graded normal domain over an algebraically closed field of positive characteristic, $Y = \text{Proj } R$. Let $I = (f_1, \dots, f_n)$ denote a homogeneous R_+ -primary ideal generated by homogeneous elements f_i of degree $d_i, i = 1, \dots, n$. Then the Hilbert-Kunz multiplicity of the ideal I equals

$$e_{HK}(I) = \frac{\deg(Y)}{2} \left(\sum_{k=1}^t r_k \nu_k^2 - \sum_{i=1}^n d_i^2 \right).$$

Here the numbers r_k and ν_k come from the strong Harder-Narasimhan filtration of the syzygy bundle $\text{Syz}(f_1^q, \dots, f_n^q)(0)$ given by the short exact sequence

$$0 \longrightarrow \text{Syz}(f_1^q, \dots, f_n^q)(0) \longrightarrow \bigoplus_{i=1}^n \mathcal{O}(-qd_i) \xrightarrow{f_1^q, \dots, f_n^q} \mathcal{O}_Y \longrightarrow 0.$$

This syzygy bundle is a locally free sheaf on the smooth projective curve $Y = \text{Proj } R$, and its strong Harder-Narasimhan filtration is a filtration $\mathcal{S}_1 \subset \dots \subset \mathcal{S}_t = \text{Syz}(f_1^q, \dots, f_n^q)(0)$ such that the quotients $\mathcal{S}_k/\mathcal{S}_{k-1}$ are strongly semistable, meaning that every Frobenius pull-back is semistable. Such a filtration exists for q big enough by a theorem of Langer, [8, Theorem 2.7]. Then we set $r_k = \text{rk}(\mathcal{S}_k/\mathcal{S}_{k-1})$ and $\nu_k = -\mu(\mathcal{S}_k/\mathcal{S}_{k-1})/q \deg(Y)$, where μ denotes the slope.

To define the Hilbert-Kunz multiplicity in characteristic zero we now take the right hand side of the above formula as our model.

Definition 1. Let R denote a two-dimensional normal standard-graded K -domain over an algebraically closed field K of characteristic zero. Let $I = (f_1, \dots, f_n)$ be a homogeneous R_+ -primary ideal given by homogeneous ideal generators f_i of degree d_i . Let $\mathcal{S}_1 \subset \dots \subset \mathcal{S}_t = \text{Syz}(f_1, \dots, f_n)(0)$ denote the Harder-Narasimhan filtration of the syzygy bundle on $Y = \text{Proj } R$, set $\mu_k = \mu(\mathcal{S}_k/\mathcal{S}_{k-1})$ and $r_k = \text{rk}(\mathcal{S}_k/\mathcal{S}_{k-1})$. Then the Hilbert-Kunz multiplicity of I is by definition

$$e_{HK}(I) = \frac{\deg(Y)}{2} \left(\sum_{k=1}^t r_k \left(\frac{\mu_k}{\deg(Y)} \right)^2 - \sum_{i=1}^n d_i^2 \right) = \frac{\sum_{k=1}^t r_k \mu_k^2 - \deg(Y)^2 \sum_{i=1}^n d_i^2}{2 \deg(Y)}.$$

It is easy to show that this definition does not depend on the chosen ideal generators and is therefore an invariant of the ideal, see [2, Proposition 4.9]. With this invariant we can in fact give the following Hilbert-Kunz criterion for solid closure in characteristic zero in dimension two (see Theorem 3.3):

Theorem 2. Let K denote an algebraically closed field of characteristic zero, let R denote a standard-graded two-dimensional normal K -domain. Let I be a homogeneous R_+ -primary ideal and let f denote a homogeneous element. Then f is contained in the solid closure, $f \in I^*$, if and only if $e_{HK}(I) = e_{HK}((I, f))$.

To prove this theorem it is convenient to consider more generally for a locally free sheaf \mathcal{S} on a smooth projective curve Y the expression

$$\mu_{HK}(\mathcal{S}) = \sum_{k=1}^t r_k \mu_k^2,$$

where r_k and μ_k are the ranks and the slopes of the semistable quotient sheaves in the Harder-Narasimhan filtration of \mathcal{S} . We call this number the Hilbert-Kunz slope of \mathcal{S} . With this notion the Hilbert-Kunz multiplicity of an ideal $I = (f_1, \dots, f_n)$ is related to the Hilbert-Kunz slope of the syzygy bundle by

$$e_{HK}((f_1, \dots, f_n)) = \frac{1}{2 \deg(Y)} \left(\mu_{HK}(\text{Syz}(f_1, \dots, f_n)(0)) - \mu_{HK}\left(\bigoplus_{i=1}^n \mathcal{O}(-d_i)\right) \right).$$

With this notion we will in fact prove the following theorem, which implies Theorem 2 (see Theorem 2.6).

Theorem 3. Let Y denote a smooth projective curve over an algebraically closed field of characteristic 0. Let \mathcal{S} denote a locally free sheaf on Y and let $c \in H^1(Y, \mathcal{S})$ denote a cohomology class given rise to the extension $0 \rightarrow \mathcal{S} \rightarrow \mathcal{S}' \rightarrow \mathcal{O}_Y \rightarrow 0$ and the affine-linear torsor $\mathbb{P}(\mathcal{S}'^\vee) - \mathbb{P}(\mathcal{S}^\vee)$. Then $\mathbb{P}(\mathcal{S}'^\vee) - \mathbb{P}(\mathcal{S}^\vee)$ is an affine scheme if and only if $\mu_{HK}(\mathcal{S}') < \mu_{HK}(\mathcal{S})$.

1. THE HILBERT-KUNZ SLOPE OF A VECTOR BUNDLE

We recall briefly some notions for locally free sheaves (or vector bundles), see [7] or [4]. Let Y denote a smooth projective curve over an algebraically closed field and let \mathcal{S} denote a locally free sheaf of rank r . Then $\deg(\mathcal{S}) = \deg(\wedge^r \mathcal{S})$ is called the degree of \mathcal{S} and $\mu(\mathcal{S}) = \deg(\mathcal{S})/r$ is called the slope of \mathcal{S} . If $\mu(\mathcal{T}) \leq \mu(\mathcal{S})$ holds for every locally free subsheaf $\mathcal{T} \subseteq \mathcal{S}$, then \mathcal{S} is called semistable. In general there exists the so-called Harder-Narasimhan filtration. This is a filtration of locally free subsheaves $\mathcal{S}_1 \subset \dots \subset \mathcal{S}_t = \mathcal{S}$ such that the quotient sheaves $\mathcal{S}_k/\mathcal{S}_{k-1}$ are semistable locally free sheaves with decreasing slopes $\mu_1 > \dots > \mu_t$. The Harder-Narasimhan filtration is uniquely determined by these properties. \mathcal{S}_1 is called the maximal destabilizing subsheaf, $\mu_1 = \mu_{\max}(\mathcal{S})$ is called the maximal slope of \mathcal{S} and $\mu_t = \mu_{\min}(\mathcal{S})$ is

called the minimal slope of \mathcal{S} . If $\mathcal{S} \rightarrow \mathcal{T}$ is a non-trivial sheaf homomorphism, then $\mu_{\min}(\mathcal{S}) \leq \mu_{\max}(\mathcal{T})$.

We begin with the definition of the Hilbert-Kunz slope of \mathcal{S} .

Definition 1.1. Let \mathcal{S} denote a locally free sheaf on a smooth projective curve over an algebraically closed field of characteristic 0. Let $\mathcal{S}_1 \subset \dots \subset \mathcal{S}_t = \mathcal{S}$ denote the Harder-Narasimhan filtration of \mathcal{S} , set $r_k = \text{rk}(\mathcal{S}_k/\mathcal{S}_{k-1})$ and $\mu_k = \mu(\mathcal{S}_k/\mathcal{S}_{k-1})$. We define the Hilbert-Kunz slope of \mathcal{S} by

$$\mu_{HK}(\mathcal{S}) = \sum_{k=1}^t r_k \mu_k^2 = \sum_{k=1}^t \frac{\deg(\mathcal{S}_k/\mathcal{S}_{k-1})^2}{r_k}.$$

The only justification for considering this number is Theorem 3.3 below. We gather together some properties of this notion in the following proposition.

Proposition 1.2. *Let \mathcal{S} denote a locally free sheaf on a smooth projective curve over an algebraically closed field of characteristic 0. Then the following hold true.*

- (i) *If \mathcal{S} is semistable, then $\mu_{HK}(\mathcal{S}) = \deg(\mathcal{S})^2 / \text{rk}(\mathcal{S})$.*
- (ii) *Let $\mathcal{T} \subset \mathcal{S}$ denote a locally free subsheaf occurring in the Harder-Narasimhan filtration of \mathcal{S} . Then $\mu_{HK}(\mathcal{S}) = \mu_{HK}(\mathcal{T}) + \mu_{HK}(\mathcal{S}/\mathcal{T})$.*
- (iii) *We have $\mu_{HK}(\mathcal{S} \oplus \mathcal{T}) = \mu_{HK}(\mathcal{S}) + \mu_{HK}(\mathcal{T})$.*
- (iv) *$\mu_{HK}(\mathcal{S}) = \mu_{HK}(\mathcal{S}^\vee)$.*
- (v) *Let \mathcal{L} denote an invertible sheaf. Then*

$$\mu_{HK}(\mathcal{S} \otimes \mathcal{L}) = \mu_{HK}(\mathcal{S}) + 2 \deg(\mathcal{S}) \deg(\mathcal{L}) + \text{rk}(\mathcal{S}) \deg(\mathcal{L})^2.$$
- (vi) *Let $\varphi : Y' \rightarrow Y$ denote a finite morphism between smooth projective curves of degree n . Then $\mu_{HK}(\varphi^*(\mathcal{S})) = n^2 \mu_{HK}(\mathcal{S})$.*

Proof. (i) and (ii) are clear from the definition. (iii). The maximal destabilizing subsheaf of $\mathcal{S} \oplus \mathcal{T}$ is either $\mathcal{S}_1 \oplus 0$, $0 \oplus \mathcal{T}_1$ or $\mathcal{S}_1 \oplus \mathcal{T}_1$. Hence the result follows from (ii) by induction on the rank of $\mathcal{S} \oplus \mathcal{T}$.

(iv). Let $0 = \mathcal{S}_0 \subset \mathcal{S}_1 \subset \dots \subset \mathcal{S}_t = \mathcal{S}$ denote the Harder-Narasimhan filtration of \mathcal{S} . Set $\mathcal{Q}_k = \mathcal{S}/\mathcal{S}_k$. This gives a filtration $0 \subset \mathcal{Q}_{t-1}^\vee \subset \dots \subset \mathcal{Q}_1^\vee \subset \mathcal{Q}_0^\vee = \mathcal{S}^\vee$. From $0 \rightarrow \mathcal{S}_k/\mathcal{S}_{k-1} \rightarrow \mathcal{S}/\mathcal{S}_{k-1} \rightarrow \mathcal{S}/\mathcal{S}_k \rightarrow 0$ we get $0 \rightarrow \mathcal{Q}_k^\vee \rightarrow \mathcal{Q}_{k-1}^\vee \rightarrow \mathcal{Q}_{k-1}^\vee/\mathcal{Q}_k^\vee \cong (\mathcal{S}_k/\mathcal{S}_{k-1})^\vee \rightarrow 0$. Hence the filtration is the Harder-Narasimhan filtration of \mathcal{S}^\vee and the result follows from $\mu(\mathcal{Q}_{k-1}^\vee/\mathcal{Q}_k^\vee) = -\mu(\mathcal{S}_k/\mathcal{S}_{k-1})$.

(v). The Harder-Narasimhan filtration of $\mathcal{S} \otimes \mathcal{L}$ is $\mathcal{S}_1 \otimes \mathcal{L} \subset \dots \subset \mathcal{S}_t \otimes \mathcal{L}$ and $\mu(\mathcal{S}_k \otimes \mathcal{L}/\mathcal{S}_{k-1} \otimes \mathcal{L}) = \mu((\mathcal{S}_k/\mathcal{S}_{k-1}) \otimes \mathcal{L}) = \mu(\mathcal{S}_k/\mathcal{S}_{k-1}) + \mu(\mathcal{L})$. Therefore

$$\begin{aligned} \mu_{HK}(\mathcal{S} \otimes \mathcal{L}) &= \sum_{k=1}^t r_k \mu_k(\mathcal{S} \otimes \mathcal{L})^2 \\ &= \sum_{k=1}^t r_k (\mu_k + \deg(\mathcal{L}))^2 \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=1}^t r_k (\mu_k^2 + 2\mu_k \deg(\mathcal{L}) + \deg(\mathcal{L})^2) \\
&= \mu_{HK}(\mathcal{S}) + 2 \deg(\mathcal{L}) \sum_{k=1}^t r_k \mu_k + \deg(\mathcal{L})^2 \sum_{k=1}^t r_k.
\end{aligned}$$

This is the stated result, since $\deg(\mathcal{S}) = \sum_{k=1}^t r_k \mu_k$ and $\mathrm{rk}(\mathcal{S}) = \sum_{k=1}^t r_k$.

(vi). The pull-back of a semistable sheaf under a separable morphism is again semistable, and the pull-back of the Harder-Narasimhan filtration is the Harder-Narasimhan filtration of $\varphi^*(\mathcal{S})$. Hence the result follows from $\deg(\varphi^*(\mathcal{S})) = n \deg(\mathcal{S})$. \square

Lemma 1.3. *The Hilbert-Kunz multiplicity of a locally free sheaf \mathcal{S} has the property that $\mu_{HK}(\mathcal{S}) \geq \deg(\mathcal{S})^2 / \mathrm{rk}(\mathcal{S})$, and equality holds if and only if \mathcal{S} is semistable.*

Proof. We have to show that

$$\sum_{k=1}^t r_k \mu_k^2 \geq \deg(\mathcal{S})^2 / \mathrm{rk}(\mathcal{S}) = (r_1 \mu_1 + \dots + r_t \mu_t)^2 / (r_1 + \dots + r_t)$$

or equivalently that

$$(r_1 + \dots + r_t) \left(\sum_{k=1}^t r_k \mu_k^2 \right) \geq (r_1 \mu_1 + \dots + r_t \mu_t)^2.$$

The left hand side is $\sum_{k=1}^t r_k^2 \mu_k^2 + \sum_{i \neq k} r_i r_k \mu_k^2$ (we sum over ordered pairs), and the right hand side is $\sum_{k=1}^t r_k^2 \mu_k^2 + \sum_{i \neq k} r_i r_k \mu_i \mu_k$. Hence left hand minus right hand is

$$\sum_{i \neq k} r_i r_k \mu_k^2 - \sum_{i \neq k} r_i r_k \mu_i \mu_k$$

So this follows from $0 \leq (\mu_i - \mu_k)^2 = \mu_i^2 + \mu_k^2 - 2\mu_i \mu_k$ for all pairs $i \neq k$. Equality holds if and only if $\mu_i = \mu_k$, but then $t = 1$ and \mathcal{S} is semistable. \square

Remark 1.4. Lemma 1.3 implies that the number $\mu_{HK}(\mathcal{S}) - \frac{\deg(\mathcal{S})^2}{\mathrm{rk}(\mathcal{S})} \geq 0$, and $= 0$ holds exactly in the semistable case. It follows from Proposition 1.2 (v) that this number is invariant under tensoring with an invertible sheaf.

Proposition 1.5. *Let \mathcal{S} and \mathcal{T} denote two locally free sheaves on Y . Then*

$$\mu_{HK}(\mathcal{S} \otimes \mathcal{T}) = \mathrm{rk}(\mathcal{T}) \mu_{HK}(\mathcal{S}) + \mathrm{rk}(\mathcal{S}) \mu_{HK}(\mathcal{T}) + 2 \deg(\mathcal{S}) \deg(\mathcal{T}).$$

Proof. Let $r_i, \mu_i, i \in I$, and $r_j, \mu_j, j \in J$, (I and J disjoint) denote the ranks and slopes occurring in the Harder-Narasimhan filtration of \mathcal{S} and \mathcal{T} respectively. It is a non-trivial fact (in characteristic zero!) that the tensor product of two semistable bundle is again semistable, see [7, Theorem 3.1.4]. From this it follows that the semistable quotients of the Harder-Narasimhan

filtration of $\mathcal{S} \otimes \mathcal{T}$ are given as $(\mathcal{S}_i/\mathcal{S}_{i-1}) \otimes (\mathcal{T}_j/\mathcal{T}_{j-1})$ of rank $r_i \cdot r_j$ and slope $\mu_i + \mu_j$. Therefore the Hilbert-Kunz slope is

$$\begin{aligned}
\mu_{HK}(\mathcal{S} \otimes \mathcal{T}) &= \sum_{i,j} r_i r_j (\mu_i + \mu_j)^2 \\
&= \sum_{i,j} r_i r_j \mu_i^2 + \sum_{i,j} r_i r_j \mu_j^2 + 2 \sum_{i,j} r_i r_j \mu_i \mu_j \\
&= (\sum_j r_j) (\sum_i r_i \mu_i^2) + (\sum_i r_i) (\sum_j r_j \mu_j^2) + 2 (\sum_i r_i \mu_i) (\sum_j r_j \mu_j) \\
&= \text{rk}(\mathcal{T}) \mu_{HK}(\mathcal{S}) + \text{rk}(\mathcal{S}) \mu_{HK}(\mathcal{T}) + 2 \deg(\mathcal{S}) \deg(\mathcal{T})
\end{aligned}$$

□

2. A HILBERT-KUNZ CRITERION FOR AFFINE TORSORS

In this section we consider a locally free sheaf \mathcal{S} on a smooth projective curve Y together with a cohomology class $c \in H^1(Y, \mathcal{S}) \cong \text{Ext}(\mathcal{O}_Y, \mathcal{S})$. Such a class gives rise to an extension $0 \rightarrow \mathcal{S} \rightarrow \mathcal{S}' \rightarrow \mathcal{O}_Y \rightarrow 0$. Of course $\deg(\mathcal{S}') = \deg(\mathcal{S})$ and $\text{rk}(\mathcal{S}') = \text{rk}(\mathcal{S}) + 1$. We shall investigate the relationship between $\mu_{HK}(\mathcal{S})$ and $\mu_{HK}(\mathcal{S}')$.

Lemma 2.1. *Let Y denote a smooth projective curve over an algebraically closed field. Let \mathcal{S} , \mathcal{T} and \mathcal{Q} denote locally free sheaves on Y . Then the following hold.*

- (i) *Let $\varphi : \mathcal{T} \rightarrow \mathcal{S}$ denote a sheaf homomorphism, $c \in H^1(Y, \mathcal{T})$ with corresponding extension \mathcal{T}' , let \mathcal{S}' denote the extension of \mathcal{S} corresponding to $\varphi(c) \in H^1(Y, \mathcal{S})$. Then there is a sheaf homomorphism $\varphi' : \mathcal{T}' \rightarrow \mathcal{S}'$ extending φ .*
- (ii) *Suppose that $0 \rightarrow \mathcal{T} \rightarrow \mathcal{S} \rightarrow \mathcal{Q} \rightarrow 0$ is a short exact sequence, and $c \in H^1(Y, \mathcal{T})$. Then $\mathcal{T}' \subseteq \mathcal{S}'$ and $\mathcal{S}'/\mathcal{T}' \cong \mathcal{S}/\mathcal{T}$.*
- (iii) *Suppose that $0 \rightarrow \mathcal{T} \rightarrow \mathcal{S} \rightarrow \mathcal{Q} \rightarrow 0$ is a short exact sequence, and $c \in H^1(Y, \mathcal{S})$. Then $\mathcal{S}' \rightarrow \mathcal{Q}' \rightarrow 0$ and $\mathcal{Q}' \cong \mathcal{S}'/\mathcal{T}$.*
- (iv) *If \mathcal{S} is semistable of degree 0 and $c \in H^1(Y, \mathcal{S})$, then also \mathcal{S}' is semistable.*

Proof. The cohomology class c is represented by the Čech cocycle $\check{c} \in H^0(U_1 \cap U_2, \mathcal{S})$, where $Y = U_1 \cup U_2$ is an affine covering. Then \mathcal{S}' arises from $\mathcal{S}'_1 = \mathcal{S}|_{U_1} \oplus \mathcal{O}$ and $\mathcal{S}'_2 = \mathcal{S}|_{U_2} \oplus \mathcal{O}$ by glueing $\mathcal{S}'_1|_{U_1 \cap U_2} \cong \mathcal{S}'_2|_{U_1 \cap U_2}$ via $(s, t) \mapsto (s + t\check{c}, t)$. The natural mappings $\mathcal{T}'_i \rightarrow \mathcal{S}'_i$, $i = 1, 2$, glue together to a morphism $\mathcal{T}' \rightarrow \mathcal{S}'$. The injectivity and surjectivity transfer from φ to φ' , since these are local properties. (ii) and (iii) then follow from suitable diagrams.

(iv). Suppose that $\mathcal{F} \subseteq \mathcal{S}'$ is a semistable subsheaf of positive slope. Then the induced mapping $\mathcal{F} \rightarrow \mathcal{O}$ is trivial and therefore $\mathcal{F} \subseteq \mathcal{S}$, which contradicts the semistability of \mathcal{S} . □

Let $\mathcal{S}_1 \subset \dots \subset \mathcal{S}_t = \mathcal{S}$ denote the Harder-Narasimhan filtration of \mathcal{S} and $c \in H^1(Y, \mathcal{S})$. If the image of c in $H^1(Y, \mathcal{S}/\mathcal{S}_{t-1})$ is zero, then c stems from a class $c_{t-1} \in H^1(Y, \mathcal{S}_{t-1})$. So we find inductively a class $c_n \in H^1(Y, \mathcal{S}_n)$ mapping to c and such that the image in $H^1(Y, \mathcal{S}_n/\mathcal{S}_{n-1})$ is not zero (or c itself is 0). This yields extensions \mathcal{S}'_k of \mathcal{S}_k for $k \geq n$. It is crucial for the behavior of \mathcal{S}' whether $\mu(\mathcal{S}_n/\mathcal{S}_{n-1}) \geq 0$ or < 0 . The following Proposition deals with the case $\mu(\mathcal{S}_n/\mathcal{S}_{n-1}) \geq 0$.

Proposition 2.2. *Let $\mathcal{S}_1 \subset \dots \subset \mathcal{S}_t = \mathcal{S}$ be the Harder-Narasimhan filtration of \mathcal{S} and let $c \in H^1(Y, \mathcal{S})$. Let n be such that the image of c in $H^1(Y, \mathcal{S}_k/\mathcal{S}_{k-1})$ is 0 for $k > n$ but such that the image in $H^1(Y, \mathcal{S}_n/\mathcal{S}_{n-1})$ is $\neq 0$. Suppose that $\mu(\mathcal{S}_n/\mathcal{S}_{n-1}) \geq 0$. Let i be the biggest number such that $\mu(\mathcal{S}_i/\mathcal{S}_{i-1}) \geq 0$ (hence $n \leq i$).*

(i) *Suppose that $\mu_i > 0$. Then the Harder-Narasimhan filtration of \mathcal{S}' is*

$$\mathcal{S}_1 \subset \dots \subset \mathcal{S}_i \subset \mathcal{S}'_i \subset \mathcal{S}'_{i+1} \subset \dots \subset \mathcal{S}'.$$

(ii) *Suppose that $\mu_i = 0$. Then the Harder-Narasimhan filtration of \mathcal{S}' is*

$$\mathcal{S}_1 \subset \dots \subset \mathcal{S}_{i-1} \subset \mathcal{S}'_i \subset \mathcal{S}'_{i+1} \subset \dots \subset \mathcal{S}'.$$

Proof. (i). The quotients of the filtration are $\mathcal{S}_k/\mathcal{S}_{k-1}$, $k \leq i$, which have positive slope, $\mathcal{S}'_i/\mathcal{S}_i \cong \mathcal{O}_Y$, and $\mathcal{S}'_k/\mathcal{S}'_{k-1} \cong \mathcal{S}_k/\mathcal{S}_{k-1}$ (Lemma 2.1(ii)) for $k > i$, which have negative slope. These quotients are all semistable and the slope numbers are decreasing.

(ii). The quotients $\mathcal{S}_k/\mathcal{S}_{k-1}$ are semistable with decreasing positive slopes for $k = 1, \dots, i-1$. The quotients $\mathcal{S}'_k/\mathcal{S}'_{k-1} \cong \mathcal{S}_k/\mathcal{S}_{k-1}$ are semistable with decreasing negative slopes for $k = i+1, \dots, t$. The quotient $\mathcal{S}'_i/\mathcal{S}_{i-1}$ is isomorphic to $(\mathcal{S}_i/\mathcal{S}_{i-1})'$ by Lemma 2.1(iii), hence semistable of degree 0 by Lemma 2.1(iv). \square

In the rest of this section we study the remaining case, that $\mu(\mathcal{S}_n/\mathcal{S}_{n-1}) < 0$. In this case it is not possible to describe the Harder-Narasimhan filtration of \mathcal{S}' explicitly. However we shall see that in this case the Hilbert-Kunz slope of \mathcal{S}' is smaller than the Hilbert-Kunz slope of \mathcal{S} . We need the following two lemmata.

Lemma 2.3. *Let \mathcal{T} denote a locally free sheaf on Y with Harder-Narasimhan filtration \mathcal{T}_k , $\mu_k = \mu(\mathcal{T}_k/\mathcal{T}_{k-1})$ and $r_k = \text{rk}(\mathcal{T}_k/\mathcal{T}_{k-1})$. Let*

$$(\tau_i) = (\mu_1, \dots, \mu_1, \mu_2, \dots, \mu_2, \mu_3, \dots, \mu_{t-1}, \mu_t, \dots, \mu_t)$$

denote the slopes where each μ_k occurs r_k -times. Let $\mathcal{S} \subseteq \mathcal{T}$ denote a locally free subsheaf of rank r and let σ_i , $i = 1, \dots, r$ denote the corresponding numbers for \mathcal{S} . Then $\sigma_i \leq \tau_i$ for $i = 1, \dots, r$.

Moreover, if \mathcal{S} is saturated (meaning that the quotient sheaf is locally free) and if no subsheaf \mathcal{S}_j of the Harder-Narasimhan filtration of \mathcal{S} occurs in the Harder-Narasimhan filtration of \mathcal{T} , then $\sigma_i \leq \tau_{i+1}$ for $i = 1, \dots, r$.

Proof. Let $i, i = 1, \dots, r$ be given and let j be such that $\text{rk}(\mathcal{S}_{j-1}) < i \leq \text{rk}(\mathcal{S}_j)$, hence $\sigma_i = \mu_j(\mathcal{S}) = \mu(\mathcal{S}_j/\mathcal{S}_{j-1})$. We may assume that $i = \text{rk}(\mathcal{S}_j)$. Let k be such that $\text{rk}(\mathcal{T}_{k-1}) < i \leq \text{rk}(\mathcal{T}_k)$. Therefore $\mathcal{S}_j \not\subseteq \mathcal{T}_{k-1}$, and the induced morphism $\mathcal{S}_j \rightarrow \mathcal{T}/\mathcal{T}_{k-1}$ is not trivial. Hence $\sigma_i = \mu_j(\mathcal{S}) = \mu_{\min}(\mathcal{S}_j) \leq \mu_{\max}(\mathcal{T}/\mathcal{T}_{k-1}) = \mu_k(\mathcal{T}) = \tau_i$.

Now suppose that $\sigma_i > \tau_{i+1}$. Then necessarily $\sigma_i > \sigma_{i+1}$ and $\tau_i > \tau_{i+1}$ by what we have already proven. Therefore $i = \text{rk}(\mathcal{S}_j) = \text{rk}(\mathcal{T}_k)$. If $\mathcal{S}_j \subseteq \mathcal{T}_k$, then they are equal, since both sheaves are saturated of the same rank, but this is excluded by the assumptions. Hence $\mathcal{S}_j \not\subseteq \mathcal{T}_k$ and $\mathcal{S}_j \rightarrow \mathcal{T}/\mathcal{T}_k$ is non-trivial. Therefore $\sigma_i = \mu_{\min}(\mathcal{S}_j) \leq \mu_{\max}(\mathcal{T}/\mathcal{T}_k) = \mu_{k+1}(\mathcal{T}) = \tau_{i+1}$. \square

Remark 2.4. If the numbers τ_i are given as in the previous lemma, then $\deg(\mathcal{T}) = \sum_i \tau_i$ and $\mu_{HK}(\mathcal{T}) = \sum_i \tau_i^2$.

Lemma 2.5. Let $\alpha_1 \leq \dots \leq \alpha_r$ and $\beta_1 \leq \dots \leq \beta_{r+1}$ denote positive real numbers such that $\alpha_i \geq \beta_{i+1}$ for $i = 1, \dots, r$ and $\sum_{i=1}^r \alpha_i = \sum_{i=1}^{r+1} \beta_i$. Then $\sum_{i=1}^{r+1} \beta_i^2 \leq \sum_{i=1}^r \alpha_i^2$ and equality holds if and only if $\alpha_i = \beta_{i+1}$.

Proof. Let $\alpha_i = \beta_{i+1} + \delta_i$, $\delta_i \geq 0$. From $\sum_{i=1}^r \alpha_i = \sum_{i=1}^r \delta_i + \sum_{i=1}^r \beta_{i+1} = \sum_{i=1}^{r+1} \beta_i$ we get $\beta_1 = \sum_{i=1}^r \delta_i$ ($\leq \beta_2$). The quadratic sums are

$$\sum_{i=1}^r \alpha_i^2 = \sum_{i=2}^{r+1} \beta_i^2 + \sum_{i=1}^r \delta_i^2 + 2 \sum_{i=1}^r \delta_i \beta_{i+1}$$

and

$$\sum_{i=1}^{r+1} \beta_i^2 = \left(\sum_{i=1}^r \delta_i \right)^2 + \sum_{i=2}^{r+1} \beta_i^2 = 2 \sum_{i < j} \delta_i \delta_j + \sum_{i=1}^r \delta_i^2 + \sum_{i=2}^{r+1} \beta_i^2.$$

So we have to show that $\sum_{i < j} \delta_i \delta_j \leq \sum_{j=1}^r \delta_j \beta_{j+1}$. But this is clear from $\sum_{i < j} \delta_j \leq \sum_{j=1}^r \delta_j \leq \beta_2 \leq \beta_{i+1}$ for all $i = 1, \dots, r$. Equality holds if and only if $\delta_i = 0$. \square

A cohomology class $H^1(Y, \mathcal{S})$ corresponds to a geometric \mathcal{S} -torsor $T \rightarrow Y$. This is an affine-linear bundle on which \mathcal{S} acts transitively. A geometric realization is given as $T = \mathbb{P}(\mathcal{S}^\vee) - \mathbb{P}(\mathcal{S}^\vee)$. The global cohomological properties of this torsor are related to the Hilbert-Kunz slope in the following way.

Theorem 2.6. Let Y denote a smooth projective curve over an algebraically closed field of characteristic 0. Let \mathcal{S} denote a locally free sheaf on Y and let $c \in H^1(Y, \mathcal{S})$ denote a cohomology class given rise to the extension $0 \rightarrow \mathcal{S} \rightarrow \mathcal{S}' \rightarrow \mathcal{O}_Y \rightarrow 0$ and the affine-linear torsor $\mathbb{P}(\mathcal{S}'^\vee) - \mathbb{P}(\mathcal{S}^\vee)$. Then the following are equivalent.

- (i) There exists a locally free quotient $\varphi : \mathcal{S} \rightarrow \mathcal{Q} \rightarrow 0$ such that $\mu_{\max}(\mathcal{Q}) < 0$ and the image $\varphi(c) \in H^1(Y, \mathcal{Q})$ is non-trivial.
- (ii) The torsor $\mathbb{P}(\mathcal{S}'^\vee) - \mathbb{P}(\mathcal{S}^\vee)$ is an affine scheme.
- (iii) The Hilbert-Kunz slope drops, that is $\mu_{HK}(\mathcal{S}') < \mu_{HK}(\mathcal{S})$.

Proof. The equivalence (i) \Leftrightarrow (ii) was shown in [3, Theorem 2.3]. The implication (iii) \Rightarrow (i) follows from Proposition 2.2: for if (i) does not hold, then we are in the situation of Proposition 2.2 that $\mu(\mathcal{S}_n/\mathcal{S}_{n-1}) \geq 0$. The explicit description of the Harder-Narasimhan filtration of \mathcal{S}' gives in both cases that $\mu_{HK}(\mathcal{S}') = \mu_{HK}(\mathcal{S})$.

So suppose that (i) holds. This means that there exists a subsheaf $\mathcal{S}_n \subseteq \mathcal{S}$ occurring in the Harder-Narasimhan filtration of \mathcal{S} such that c stems from $c_n \in H^1(Y, \mathcal{S}_n)$ and such that its image in $H^1(Y, \mathcal{S}/\mathcal{S}_{n-1})$ is non-trivial with $\mu_{\max}(\mathcal{S}/\mathcal{S}_{n-1}) = \mu(\mathcal{S}_n/\mathcal{S}_{n-1}) = \mu_n < 0$.

Let $\mathcal{T}_1 \subset \dots \subset \mathcal{T}_t = \mathcal{S}'$ denote the Harder-Narasimhan filtration of \mathcal{S}' with slopes $\mu_k = \mu(\mathcal{T}_k/\mathcal{T}_{k-1})$ and ranks $r_k = \text{rk}(\mathcal{T}_k/\mathcal{T}_{k-1})$. Suppose that the maximal slope $\mu(\mathcal{T}_1)$ is positive. Then the induced mapping $\mathcal{T}_1 \rightarrow \mathcal{S}'/\mathcal{S} \cong \mathcal{O}_Y$ is trivial, and $\mathcal{T}_1 \subseteq \mathcal{S}$. This is then also the maximal destabilizing subsheaf of \mathcal{S} , since $\mu_{\max}(\mathcal{S}) \leq \mu_{\max}(\mathcal{S}') = \mu(\mathcal{T}_1)$. Therefore $\mu_{HK}(\mathcal{S}) = \mu_{HK}(\mathcal{T}_1) + \mu_{HK}(\mathcal{S}/\mathcal{T}_1)$ and $\mu_{HK}(\mathcal{S}') = \mu_{HK}(\mathcal{T}_1) + \mu_{HK}(\mathcal{S}'/\mathcal{T}_1)$ by Proposition 1.2(ii). Since $\mathcal{S}'/\mathcal{T}_1$ is the extension of $\mathcal{S}/\mathcal{T}_1$ defined by the image of the cohomology class in $H^1(Y, \mathcal{S}/\mathcal{T}_1)$ (Lemma 2.1(iii)), we may mod out \mathcal{T}_1 . Note that this does not change the condition in (i). Hence we may assume inductively that $\mu_{\max}(\mathcal{S}) \leq 0$ and $\mu_{\max}(\mathcal{S}') \leq 0$.

Now suppose that \mathcal{T}_1 has degree 0. Again, if $\mathcal{T}_1 \subseteq \mathcal{S}$, then this is also the maximal destabilizing subsheaf of \mathcal{S} , and we can mod out \mathcal{T}_1 as before. So suppose that $\mathcal{T}_1 \rightarrow \mathcal{O}_Y$ is non-trivial. Then this mapping is surjective, let $\mathcal{K} \subset \mathcal{S}$ denote the kernel. This means that the extension defined by $c \in H^1(Y, \mathcal{S})$ comes from the extension given by $0 \rightarrow \mathcal{K} \rightarrow \mathcal{T}_1 \rightarrow \mathcal{O}_Y \rightarrow 0$, and $\tilde{c} \in H^1(Y, \mathcal{K})$. \mathcal{K} is semistable, since its degree is 0 and $\mu_{\max}(\mathcal{S}) \leq 0$. But then the image of c is 0 in every quotient sheaf of \mathcal{S} with negative maximal slope, which contradicts the assumptions. Therefore we may assume that $\mu_{\max}(\mathcal{S}') < 0$.

We want to apply Lemma 2.3 to $\mathcal{S} \subset \mathcal{S}' = \mathcal{T}$. Assume that \mathcal{S} and \mathcal{S}' have a common subsheaf occurring in both Harder-Narasimhan filtrations. Then they have the same maximal destabilizing subsheaf $\mathcal{F} = \mathcal{S}_1 = \mathcal{T}_1$, which has negative degree. If c comes from $\tilde{c} \in H^1(Y, \mathcal{F})$, then $\mathcal{F} \subset \mathcal{F}' \subseteq \mathcal{S}'$ and $\mu(\mathcal{F}) = \deg(\mathcal{F})/\text{rk}(\mathcal{F}) < \deg(\mathcal{F})/(\text{rk}(\mathcal{F}) + 1) = \mu(\mathcal{F}')$, which contradicts the maximality of \mathcal{F} . Hence the image of c in $H^1(Y, \mathcal{S}/\mathcal{F})$ is not zero and we can mod out \mathcal{F} as before.

Therefore we may assume that \mathcal{S} and \mathcal{S}' do not have any common subsheaf in their Harder-Narasimhan filtrations. Then Lemma 2.3 yields that $\sigma_i \leq \tau_{i+1}$, and all these numbers are ≤ 0 and moreover $\tau_i < 0$. Lemma 2.5 applied to $\alpha_i = -\sigma_i$ and $\beta_i = -\tau_i$ yields that $\sum_{i=1}^r \sigma_i^2 \geq \sum_{i=1}^{r+1} \tau_i^2$, and $>$ holds since $\tau_1 \neq 0$. \square

Remark 2.7. Suppose that \mathcal{S} is a semistable locally free sheaf of negative degree, and let $c \in H^1(Y, \mathcal{S})$ with corresponding extension \mathcal{S}' . Then Theorem

2.6 together with Lemma 1.3 yield the inequalities

$$\frac{\deg(\mathcal{S})^2}{r+1} \leq \mu_{HK}(\mathcal{S}') \leq \frac{\deg(\mathcal{S})^2}{r}.$$

If \mathcal{S}' is also semistable, then we have equality on the left.

3. A HILBERT-KUNZ CRITERION FOR SOLID CLOSURE

We come now back to our original setting of interest, that of a two-dimensional normal standard-graded domain R over an algebraically closed field K . A homogeneous R_+ -primary ideal $I = (f_1, \dots, f_n)$ gives rise to the syzygy bundle $\text{Syz}(f_1, \dots, f_n)(0)$ on $Y = \text{Proj } R$ defined by the presenting sequence

$$0 \longrightarrow \text{Syz}(f_1, \dots, f_n)(m) \longrightarrow \bigoplus_{i=1}^n \mathcal{O}_Y(m - d_i) \xrightarrow{f_1, \dots, f_n} \mathcal{O}_Y(m) \longrightarrow 0.$$

Another homogeneous element f of degree m yields an extension

$$0 \longrightarrow \text{Syz}(f_1, \dots, f_n)(m) \longrightarrow \text{Syz}(f_1, \dots, f_n, f)(m) \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

which corresponds to the cohomology class $\delta(f) \in H^1(Y, \text{Syz}(f_1, \dots, f_n)(m))$ coming from the presenting sequence via the connecting homomorphism

$$\delta : H^0(Y, \mathcal{O}_Y(m)) = R_m \rightarrow H^1(Y, \text{Syz}(f_1, \dots, f_n)(m)).$$

The Hilbert-Kunz multiplicities of the ideals and the Hilbert-Kunz slopes of the syzygy bundles are related in the following way.

Lemma 3.1. *Let K denote an algebraically closed field of characteristic 0. Let R denote a standard-graded two-dimensional normal K -domain, $Y = \text{Proj } R$. Let I be a homogeneous R_+ -primary ideal and let f denote a homogeneous element of degree m . Then the Hilbert-Kunz multiplicities $e_{HK}(I) = e_{HK}((I, f))$ are equal if and only if the Hilbert-Kunz slopes of the corresponding syzygies bundles $\mu_{HK}(\text{Syz}(f_1, \dots, f_n)(m)) = \mu_{HK}(\text{Syz}(f_1, \dots, f_n, f)(m))$ are equal.*

Proof. Let μ_k and r_k ($\tilde{\mu}_k$ and \tilde{r}_k) denote the ranks and the slopes in the Harder-Narasimhan filtration of $\text{Syz}(f_1, \dots, f_n)(0)$ (of $\text{Syz}(f_1, \dots, f_n, f)(0)$ respectively). For the Hilbert-Kunz multiplicities of the ideals (f_1, \dots, f_n) and (f_1, \dots, f_n, f) we have to compare

$$e_{HK}(I) = \frac{1}{2 \deg(Y)} \left(\sum_{k=1}^t r_k \mu_k^2 - \deg(Y)^2 \sum_{i=1}^n d_i^2 \right)$$

and

$$e_{HK}((I, f)) = \frac{1}{2 \deg(Y)} \left(\sum_{k=1}^{\tilde{t}} \tilde{r}_k \tilde{\mu}_k^2 - \deg(Y)^2 (m^2 + \sum_{i=1}^n d_i^2) \right).$$

The extension defined by $c = \delta(f) \in H^1(Y, \text{Syz}(f_1, \dots, f_n)(m))$ is

$$0 \longrightarrow \mathcal{S} = \text{Syz}(f_1, \dots, f_n)(m) \longrightarrow \mathcal{S}' = \text{Syz}(f_1, \dots, f_n, f)(m) \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

and the Hilbert-Kunz slopes of these sheaves are due to Proposition 1.2 (v) (since $\deg(\text{Syz}(f_1, \dots, f_n)(0)) = -\deg(Y) \sum_{i=1}^n d_i$)

$$\mu_{HK}(\mathcal{S}) = \sum_{k=1}^t r_k \mu_k^2 + 2(-\sum_{i=1}^n d_i \deg(Y))m \deg(Y) + (n-1)m^2 \deg(Y)^2$$

and $\mu_{HK}(\mathcal{S}') =$

$$\begin{aligned} &= \sum_{k=1}^{\tilde{t}} \tilde{r}_k \tilde{\mu}_k^2 + 2(-(\sum_{i=1}^n d_i + m) \deg(Y))m \deg(Y) + nm^2 \deg(Y)^2 \\ &= \sum_{k=1}^{\tilde{t}} \tilde{r}_k \tilde{\mu}_k^2 - 2(\sum_{i=1}^n d_i)m \deg(Y)^2 + (n-1)m^2 \deg(Y)^2 - m^2 \deg(Y)^2. \end{aligned}$$

So the difference is in both cases (up to the factor $1/2 \deg(Y)$)

$$\sum_{k=1}^{\tilde{t}} \tilde{r}_k \tilde{\mu}_k^2 - \sum_{k=1}^t r_k \mu_k^2 - m^2 \deg(Y)^2.$$

Therefore $e_{HK}(I) = e_{HK}((I, f))$ if and only if

$$\mu_{HK}(\text{Syz}(f_1, \dots, f_n)(m)) = \mu_{HK}(\text{Syz}(f_1, \dots, f_n, f)(m)).$$

□

Remark 3.2. Let $0 \rightarrow \mathcal{S} \rightarrow \mathcal{T} \rightarrow \mathcal{Q} \rightarrow 0$ denote a short exact sequence of locally free sheaves. Then the alternating sum of the Hilbert-Kunz slopes, that is $\mu_{HK}(\mathcal{S}) - \mu_{HK}(\mathcal{T}) + \mu_{HK}(\mathcal{Q})$ does not change when we tensor the sequence with an invertible sheaf. This follows from Proposition 1.2(v). For an extension $0 \rightarrow \mathcal{S} \rightarrow \mathcal{S}' \rightarrow \mathcal{O}_Y \rightarrow 0$ this number is ≥ 0 by Theorem 2.6, and we suspect that this is true in general. From the presenting sequence $0 \rightarrow \text{Syz}(f_1, \dots, f_n)(0) \rightarrow \bigoplus_{i=1}^n \mathcal{O}(-d_i) \rightarrow \mathcal{O}_Y \rightarrow 0$ it follows via $e_{HK}(I) = \frac{1}{2 \deg(Y)} (\mu_{HK}(\text{Syz}(f_1, \dots, f_n)(0)) - \mu_{HK}(\bigoplus_{i=1}^n \mathcal{O}(-d_i)))$ that the Hilbert-Kunz multiplicity of an ideal is always nonnegative. In fact $I = R$ is the only ideal with $e_{HK}(I) = 0$. This follows from Theorem 3.3 below, since $1 \notin I^*$ for $I \neq R$.

We come now to the main result of this paper. Recall that the solid closure of an \mathfrak{m} -primary ideal $I = (f_1, \dots, f_n)$ in a two-dimensional normal excellent domain R is given by the condition that $f \in (f_1, \dots, f_n)^*$ if and only if $D(\mathfrak{m}) \subset \text{Spec } R[T_1, \dots, T_n]/(f_1 T_1 + \dots + f_n T_n + f)$ is not an affine scheme. In positive characteristic this is the same as tight closure, see [5, Theorem 8.6]. In the case of an R_+ -primary homogeneous ideal in a standard-graded normal K -domain this is equivalent to the property that the torsor $\mathbb{P}(\mathcal{S}'^\vee) - \mathbb{P}(\mathcal{S}^\vee)$ over the corresponding curve $Y = \text{Proj } R$ is not affine (see [1, Proposition 3.9]). This relates solid closure to the setting of the previous section.

Theorem 3.3. *Let K denote an algebraically closed field. Let R denote a standard-graded two-dimensional normal K -domain. Let I be a homogeneous R_+ -primary ideal and let f denote a homogeneous element. Then $f \in I^*$ if and only if $e_{HK}(I) = e_{HK}((I, f))$.*

Proof. If the characteristic is positive then this is a standard result from tight closure theory as mentioned in the introduction. So suppose that the characteristic is 0. Let $I = (f_1, \dots, f_n)$ be generated by homogeneous elements, and set $m = \deg(f)$. The containment in the solid closure, $f \in (f_1, \dots, f_n)^*$, is equivalent with the non-affineness of the torsor $\mathbb{P}(\mathcal{S}'^\vee) - \mathbb{P}(\mathcal{S}^\vee)$ [1, Proposition 3.9], where $\mathcal{S} = \text{Syz}(f_1, \dots, f_n)(m)$ and \mathcal{S}' is the extension given by the cohomology class $\delta(f)$. Hence the result follows from Theorem 2.6 and Lemma 3.1. \square

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